Cause and Effect of the -u^2 Term in Induced Drag[1]

Charles W. McCutchen

5213 Acacia Ave. Bethesda, Maryland 20814-2893

Abstract

The induced drag calculated from velocities in the rolled-up wake of a wing is less by $-\rho\int u^2 dA$ than the kinetic energy per unit length of flight calculated from those same velocities, suggesting a violation of conservation of energy. But the slope of the vortex-pair wake shortens its projection on the direction of flight by the amount needed to make the rate of energy gain match the drag power. This note calculates the angle of slope from the velocity components in the rolled-up wake, and gives examples of the overestimation of induced drag when the $-u^2$ term is omitted. The error increases somewat faster than the square of the quotient of lift coefficient by aspect ratio.

Key words: Induced drag; minus integral *u* squared; paradox explained

[1] This note has been unpublishble. In the words of an observer, "More gripes from McCutchen. Of course he is right, but unfortunatly it is not easy to fix this situation."

Nomenclature

A	=	a plane in the rolled up wake transverse to the direction of flight
AR	=	Aspect ratio, b^2/S
b	=	wingspan
C_L	=	lift coefficient
KE	=	kinetic energy per unit length of flight
L	=	Length of an arbitrary section of flight path
S	=	Wing area
u	=	aftward component of wake velocity along the flight path relative to fluid at infinity. It was "a" in Reference [1]
v	=	lateral component of wake velocity relative to fluid at infinity
V	=	forward speed of the wing relative to fluid at infinity
w	=	downward component of wake velocity relative to fluid at infinity
ε	=	inclination of wake axis
ρ	=	density of the fluid

1. The paradox

A coordinate system is centered on the wing of a moving airplane. If u, parallel to the flight path, v and w are rearward, lateral and downward components of flow velocity far enough behind an airplane wing that its vortex sheet has rolled up, the ideal, zero-viscosity induced drag is [1, 2],

$$D_i = (\rho/2)\int(v^2+w^2-u^2)dA. \qquad (1$$

The $-u^2$ term is negative thanks to $-\rho\int u^2 dA$, the contribution of momentum flux [1]. But even thus explained, a term that looks like energy production yet reduces drag can be disquieting. Landau and Lifshitz [3] found the $-u^2$ term, but, as others have since, omitted it from the expression for induced drag because it would be small.

Adding puzzlement is that the kinetic energy per unit length of flight

$$KE = (\rho/2)\int(v^2+w^2+u^2)dA \quad [1], \qquad (2$$

is larger than the induced drag by $\rho\int u^2 dA$ [1], suggesting a violation of energy conservation. On his page 414 Spalart [4] gives examples of the resulting confusion.

2. The paradox explained

The paradox springs from the slope of the wing's vortex wake, which makes it shorter than the distance traveled by the wing. See the Figure.

```
                                                 ∫
=================================================
<-----------------------------------------------------------------------------•
  00000                    |                     ∫           /
        00000              angle ε               ∫         /
              00000     /                        ∫       /
                    00000                        ∫     /
                          00000                  ∫    /
                                00000            ∫  /
                                      00000      ∫ /
                                            00000
                                                 ∫
```

<-------------•	An arbitrary length L of the flight path.
00000 00000	The vortex pair created during length L of the flight. The pair's length is Lcosε.
==========	The projection of the vortex pair onto the flight axis. The projection's length is $L\cos^2\varepsilon$. A plane of integration, A, is marked by integral signs.

The flight path and vortex pair, viewed from the side

The vortex pair slopes downward at an angle ε, twice the "induced angle of attack" or induced drag coefficient, so that rather than lengthen as fast as the airplane flies, the pair, moving at right angles to the axes of its vortices, lengthens at a rate $V\cos\varepsilon$, and its projection on the flight path lengthens at $V\cos^2\varepsilon$.[2] (Remember that the integrals in Eqs. 1 and 2 are both taken over a plane perpendicular to the flight path.)

The kinetic energy *created* in flying a distance L is kinetic energy created per unit length of flight, $(\rho/2)\int(v^2+w^2+u^2)dA$ times $L\cos^2\varepsilon$, and it must equal induced drag times L. Thus

$$L\cos^2\varepsilon(\rho/2)\int(v^2+w^2+u^2)dA = L(\rho/2)\int(v^2+w^2-u^2)dA, \text{ or}$$

$$\cos^2\varepsilon\int(v^2+w^2+u^2)dA = \int(v^2+w^2-u^2)dA, \qquad (3$$

hence, using $\cos^2\varepsilon \equiv 1 - \sin^2\varepsilon$,

[2] Spalart [4], working in coordinates aligned with the axis of the vortex pair, recognized that the length of the vortex wake is $\cos\varepsilon$ times the distance traveled by the wing, but he did not proceed to the product of $\cos^2\varepsilon$ by distance traveled, *i. e.,* the projection of the wake on the direction of flight.

He wrote that there was no "paradox attached to the $-u^2/2$ term."

$$(1 - \sin^2\varepsilon)\int(v^2+w^2+u^2)dA = \int(v^2+w^2-u^2)dA.$$

from which

$$-\sin^2\varepsilon\int(v^2+w^2+u^2)dA = \int(v^2+w^2-u^2)dA - \int(v^2+w^2+u^2)dA = -2\int u^2dA \quad (4$$

so

$$\sin^2\varepsilon = 2\int u^2dA/\int(v^2+w^2+u^2)dA. \quad (4a$$

3. Consequences

0mitting the $-u^2$ term overestimates the induced drag by a factor

$$\int(v^2+w^2)dA/\int(v^2+w^2-u^2)dA$$
$$= [\int(v^2+w^2+u^2)dA-\int u^2dA]/[\int(v^2+w^2+u^2)dA-2\int u^2dA]$$
$$= [1-\int u^2dA/\int(v^2+w^2+u^2)dA]/[1-2\int u^2dA/\int(v^2+w^2+u^2)dA]$$
$$= [1-(1/2)\sin^2\varepsilon]/(1-\sin^2\varepsilon). \quad (5$$

$\mathrm{Sin}^2\varepsilon$ is roughly $[4(\mathrm{lift})/(\rho V^2\pi b^2)]^2$ [5] $= [(2/\pi)(C_LS/b^2)]^2$ $= (2C_L/\pi AR)^2$, $(2/\pi)^2$ times the square of lift coefficient over aspect ratio.

The aspect ratio of the Lockheed C-5A is 8, the lift coefficent at landing is 2.8 [6], thus $\sin^2\varepsilon$ = .04965 and ε = 12.88 degrees. (Typically the high C_L does not extend over the entire wingspan, introducing errors in effective C_L and effective aspect ratio that will compensate each other to some degree.) Ignoring the $-u^2$ term overstimates the induced drag by a factor [1-(1/2).04965]/(1-.04965) = 1.026, making the overestimate 2.612%.

Now imagine the C-5A cruising at C_L = 1, thus $\sin^2\varepsilon$ = .006333 and ε = 4.564 degrees). The overestimate of induced drag is .3186%.

Now the C-5A is empty on a short, positioning flight. Its C_L is .5, $\sin^2\varepsilon$ = .001583, and ε = 2.280 degrees. The overestimate of induced drag is .07928%.

Next keep C_L at 2.8, but lower the aspect ratio to 3 as in the F-15 (reference 6, page 129). For this landing fighter $\sin^2\varepsilon$ = .3530, ε = 36.54 degrees, the overestimate of induced drag is 27.29%.

4. Conclusions

Unmasking the apparent energy paradox should quiet worries about the validity of the $-u^2$ term in Eq. 1 for induced drag. Not as negligible as generally regarded, its importance rises with high C_L and low aspect ratio.

That it has received so little attention may be because people, including myself [1] and Spalart [4], did not realize that *u* component, fore-and-aft velocities result from the circumferential motion in the tilted vorticies of the vortex pair as well as from their axial flow.[3]

5. Addendum

From Eq. 1 the induced drag is zero if $\int u^2 dA = \int (v^2+w^2) dA$. Substituting this into Eq. 4a, $\sin^2\varepsilon = (2\int u^2 dA)/(2\int u^2 dA) = 1 = \sin\varepsilon$, so $\varepsilon = 90$ degrees. The vortex pair is vertical and generates no lift. Its length is $L\cos\varepsilon = 0$, so it contains no energy, which explains the lack of induced drag and gives a second reason for zero lift.

If $\int u^2 dA > \int (v^2+w^2) dA$, $\sin^2\varepsilon > 1$, which is imposible, so the wake cannot be obeying Bernoulli's theorem. Perhaps some of its fluid went through a jet engine. The converse, $\int u^2 dA < \int (v^2+w^2) dA$, does not prove that the wake obeys Bernoulli's theorem.

References

[3] In the inviscid world of this paper the axial flows meet at the starting vortex, causing it and the adjacent part of the pair to expand in a region that slowly lengthens toward the wing.

[1] McCutchen, C. W., "Induced Drag and the Ideal Wake of a Lifting Wing, "*Journal of Aircraft,* Vol. 26, No. 5, May 1989, pp. 489-493. In this reference I wrongly stated that the reduction in lift due to the downward aim of wake suction was not accounted for in its Eq. (17).

[2] Schouten, G., "Momentum, Pressure and Energy in the Trefftz Plane." *Journal of Aircraft,* Vol. 32, No. 5, Sept.-Oct., 1995, pp. 943-954.

[3] Landau, L. D. and Lifshitz, E. M., *Fluid Dynamics,* Pergamon, London, 1959, pp. 175-176.

[4] Spalart, P. R., "On the far wake and induced drag of aircraft," *Journal of Fluid Mechanics,* Vol. 603, May 2008, pp. 413 - 430.

[5] Jones, R. T., "Wing Theory." Princeton. p. 108.

[6] Lowrey, R. O., Evolution of Transport Wings From C-130, C 141, C-5 to C-XX, *in* Evolution of Aircraft Wing Design, AIAA Synposium, Dayton, March 18-19, 1980. p. 69.

Jan. 20, 2019 1435 words